Hence

$$\int_{C_R} \frac{z\,dz}{\sqrt{(z-a)(z-b)}} = \pi(a+b)i.$$

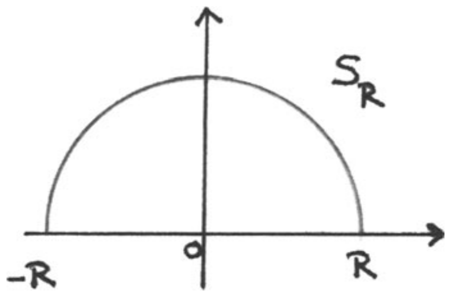
VI.2 Evaluation of Definite Integrals

Exercise VI.2.1. *Find the following integrals:*
(a) $\int_{-\infty}^{\infty} \frac{1}{x^6+1} dx = 2\pi/3.$
(b) Show that for a positive integer $n \geq 2$,

$$\int_0^\infty \frac{1}{1+x^n} dx = \frac{\pi/n}{\sin \pi/n}.$$

[Hint: Try the path from 0 to R, then from R to $Re^{2\pi i/n}$*, then back to 0, or apply a general theorem.]*

Solution. (a) Consider the contour shown on the figure, namely a symmetric segment on the real line and a semicircle in the upper half plane.



We have

$$\left| \int_{S_R} \frac{1}{1+z^6} dz \right| \leq \pi R \frac{B}{R^6}$$

for some constant B valid for all large R. This shows that the integral on the semicircle goes to zero as R tends to infinity, and by the residue formula

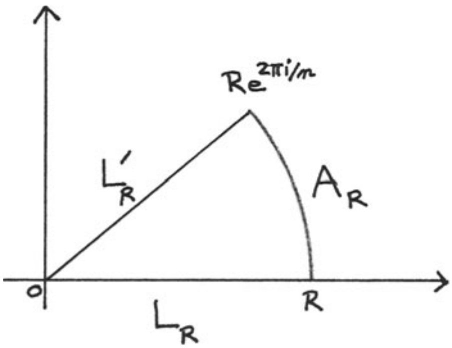
$$\int_{-\infty}^{\infty} \frac{1}{x^6+1} dx = 2\pi i \sum \text{residues of } \frac{1}{1+z^6} \text{ in the upper half plane.}$$

The poles of $1/(1+z^6)$ in the upper half plane are at the points $e^{i\pi/6}$, $e^{i\pi/2}$ and $e^{i5\pi/6}$. Moreover, these poles are simple, so we can use the derivative to find the residues. It follows that the desired integral is

$$\int_{-\infty}^{\infty} \frac{1}{1+x^6}dx = 2\pi i(\frac{e^{-5i\pi/6}}{6} + \frac{e^{-5i\pi/2}}{6} + \frac{e^{-25i\pi/2}}{6})$$

$$= \frac{\pi i}{3}\left(-\frac{\sqrt{3}}{2} - \frac{i}{2} - i - \frac{i}{2} + \frac{\sqrt{3}}{2}\right) = \frac{2\pi}{3}.$$

(b) We split the contour integral given in the hint in three parts, L_R the segment from 0 to R, A_R the arc from R to $Re^{2\pi i/n}$, and L'_R the segment from $Re^{2\pi i/n}$ to 0.



The integral on the arc tends to 0 as R becomes large because this integral is estimated by the sup norm of f multiplied by the length of the arc, and because we assume $n \geq 2$

$$\left|\int_{A_R} \frac{1}{1+z^n}dz\right| \leq R\frac{2\pi}{n}\frac{B}{R^n}.$$

The only pole of $1/(1+z^n)$ in the interior of the contour (for large R) is $e^{\pi i/n}$ and this pole is simple. The derivative shows that the residue is

$$\frac{1}{n}e^{-(n-1)\pi i/n} = \frac{-1}{n}e^{\pi i/n}.$$

Parametrizing L'_R by $te^{2\pi i/n}$ with $0 \leq t \leq R$ we find that

$$\int_{L'_R} \frac{1}{1+z^n}dz = -e^{2\pi i/n}\int_{L_R} \frac{1}{1+z^n}dz.$$

Taking the limit as $R \to \infty$ and using the residue formula we get

$$(1 - e^{2\pi i/n})\int_0^{\infty} \frac{1}{1+x^n}dx = 2\pi i(\frac{-1}{n}e^{\pi i/n}),$$

thus

$$\frac{(e^{\pi i/n} - e^{-\pi i/n})}{2i}\int_0^{\infty} \frac{1}{1+x^n}dx = \pi/n.$$

By Euler's formula we conclude that

$$\int_0^\infty \frac{1}{1+x^n}dx = \frac{\pi/n}{\sin \pi/n}.$$

Exercise VI.2.2. *Find the following integrals:*
(a) $\int_{\infty}^{\infty} \frac{x^2}{x^4+1}dx = \pi\sqrt{2}/2.$
(b) $\int_0^{\infty} \frac{x^2}{x^6+1}dx = \pi/6.$

Solution. (a) Let $f(z) = z^2/(1+z^4)$. To use the contour given in the text, i.e., a segment on the real line and a semicircle in the upper half plane (see the first figure of the preceding exercise) we must show that f decreases rapidly at infinity. There exists a constant B such that for all large R we have

$$|f(z)| \leq B\frac{R^2}{R^4} = \frac{B}{R^2} \quad \text{whenever } |z| = R.$$

The integral on the semicircle is estimated by the sup norm of f multiplied by the length of the semicircle. Hence the integral on the semicircle is bounded by $\pi R(B/R^2) = \pi B/R$, and therefore this integral tends to 0 as R tends to infinity. So

$$\int_{-\infty}^{\infty} f(x)dx = 2\pi i \sum \text{residues of } \frac{z^2}{1+z^4} \text{ in the upper half plane.}$$

The function $f(z)$ has two simple poles in the upper half plane at $e^{\pi i/4}$ and $e^{3\pi i/4}$. Using the derivative of the denominator and the fact that the numerator is entire, we find that the residues are

$$\frac{(e^{\pi i/4})^2}{4e^{3\pi i/4}} \quad \text{and} \quad \frac{(e^{3\pi i/4})^2}{4e^{9\pi i/4}},$$

respectively. Hence

$$\begin{aligned}\int_{-\infty}^{\infty} f(x)dx &= 2\pi i \left(\frac{e^{\pi i/2}}{4e^{3\pi i/4}} + \frac{e^{3\pi i/2}}{4e^{\pi i/4}}\right) \\ &= \frac{\pi i}{2}(e^{-\pi i/4} + e^{5\pi i/4}) \\ &= \frac{\pi i}{2}\left(-2i\frac{\sqrt{2}}{2}\right) = \frac{\pi\sqrt{2}}{2}.\end{aligned}$$

(b) Let $f(z) = z^2/(1+z^6)$. The function f is even, so

$$\int_{-\infty}^{\infty} f(x)dx = 2\int_0^{\infty} f(x)dx,$$

and we are reduced to computing the integral of f over the whole real line. Arguing like in (a) we see that we can use the same contour, hence

$$\int_{-\infty}^{\infty} f(x)dx = 2\pi i \sum \text{residues of } \frac{z^2}{1+z^6} \text{ in the upper half plane.}$$

The poles of f are described in part (a) of Exercise 1. Taking into account that z^2 is entire we can compute the residues at the poles and obtain

$$\begin{aligned}\int_{-\infty}^{\infty} f(x)dx &= 2\pi i\left(\frac{e^{2\pi i/6}}{6e^{5\pi i/6}}+\frac{e^{2\pi i/2}}{6e^{5\pi i/2}}+\frac{e^{10\pi i/6}}{6e^{25\pi i/6}}\right)\\ &= \frac{\pi i}{3}(e^{-\pi i/2}+e^{-3\pi i/2}+e^{-\pi i/2})\\ &= \frac{\pi i}{3}(-i+i-i)=\frac{\pi}{3}.\end{aligned}$$

The above observation implies that

$$\int_{0}^{\infty} f(x)dx = \frac{\pi}{6},$$

as was to be shown.

Exercise VI.2.3. *Show that*

$$\int_{-\infty}^{\infty}\frac{x-1}{x^5-1}dx = \frac{4\pi}{5}\sin\frac{2\pi}{5}.$$

Solution. Let $f(z)=(z-1)/(z^5-1)$. Then there exists a positive constant B such that for all large R we have

$$|f(z)| \le B\frac{R^2}{R^5} = \frac{B}{R^3}$$

whenever $|z|=R$. The same argument as in Exercise 1 (a) shows that we can use the same contour as this exercise, therefore

$$\int_{-\infty}^{\infty}\frac{x-1}{x^5-1}dx = 2\pi i\sum \text{residues of } \frac{z-1}{z^5-1} \text{ in the upper half plane.}$$

The simple poles of f in the upper half plane are at the points $e^{2\pi i/5}$ and $e^{4\pi i/5}$, so the residues at these points are

$$\frac{e^{2\pi i/5}-1}{5(e^{2\pi i/5})^4} = \frac{e^{4\pi i/5}-e^{2\pi i/5}}{5} \quad\text{and}\quad \frac{e^{4\pi i/5}-1}{5(e^{4\pi i/5})^4} = \frac{e^{8\pi i/5}-e^{4\pi i/5}}{5}.$$

Therefore

$$\begin{aligned}\int_{-\infty}^{\infty}\frac{x-1}{x^5-1}dx &= \frac{2\pi i}{5}(e^{4\pi i/5}-e^{2\pi i/5}+e^{8\pi i/5}-e^{4\pi i/5})\\ &= \frac{2\pi i}{5}(-e^{2\pi i/5}+e^{-2\pi i/5})\\ &= \frac{2\pi i}{5}2i\sin(2\pi/5)\\ &= \frac{2\pi i}{5} - 2i\sin(-2\pi/5)\\ &= \frac{4\pi}{5}\sin(2\pi/5)\end{aligned}$$

as was to be shown.

Exercise VI.2.4. *Evaluate*

$$\int_\gamma \frac{e^{-z^2}}{z^2}dz,$$

where γ *is:*
(a) the square with vertices $1+i, -1+i, -1-i, 1-i$.
(b) the ellipse defined by the equation

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1.$$

(The answer is 0 in both cases.)

Solution. The only singularity of the function e^{-z^2}/z^2 is at the origin. The power series expansion for the exponential gives

$$\frac{e^{-z^2}}{z^2}=\frac{1}{z^2}-1+\frac{z^2}{2!}-\frac{z^4}{3!}$$

so 0 is a pole of order 2. From the above expression we also see that the residue of e^{-z^2}/z^2 at the origin is 0. By the residue formula we conclude that the answer to (a) and (b) is 0.

Exercise VI.2.5. *(a)* $\int_{-\infty}^{\infty}\frac{e^{iax}}{x^2+1}dx=\pi e^{-a}$ *if* $a>0$.
(b) For any real number $a>0$,

$$\int_{-\infty}^{\infty}\frac{\cos x}{x^2+a^2}dx=\pi e^{-a}/a.$$

[Hint: This is the real part of the integral obtained by replacing $\cos x$ *by* e^{ix}*.]*

Solution. (a) This integral belongs to the section on Fourier transforms: We must show that $f(z)=1/(1+z^2)$ goes to 0 fast enough. There exists a constant K such that for all sufficiently large $|z|$ we have

$$|f(z)|\le\frac{K}{|z|^2},$$

so the decay assumption is satisfied and we can use the formula given in the text (Theorem 2.2)

$$\int_{-\infty}^{\infty}\frac{e^{iax}}{1+x^2}dx=2\pi i\sum \text{residues of } e^{iaz}f(z) \text{ in the upper half plane.}$$

The function f has a simple pole at i with residue $1/2i$, so

$$\int_{-\infty}^{\infty}\frac{e^{iax}}{1+x^2}dx=2\pi i\left(\frac{e^{iai}}{2i}\right)=\pi e^{-a},$$

as was to be shown.
(b) Changing variables $x=ay$ we get

$$\int_{-\infty}^{\infty}\frac{\cos x}{x^2+a^2}dx=\frac{1}{a}\int_{-\infty}^{\infty}\frac{\cos(ay)}{y^2+1}dy$$

$$= \frac{1}{a} \operatorname{Re} \left(\int_{-\infty}^{\infty} \frac{e^{iay}}{y^2+1} dy \right)$$
$$= \frac{1}{a} \pi e^{-a},$$

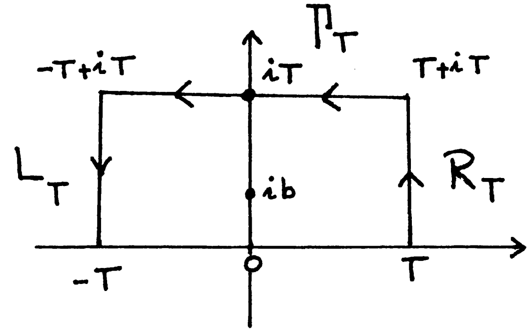
as was to be shown.

Exercise VI.2.6. *Let $a, b > 0$. Let $T \geq 2b$. Show that*

$$\left| \frac{1}{2\pi i} \int_{-T}^{T} \frac{e^{iaz}}{z - ib} dz - e^{-ba} \right| \leq \frac{1}{Ta}(1 - e^{-Ta}) + e^{-Ta}.$$

Formulate a similar estimate when $a < 0$.

Solution. Let $f(z) = e^{iaz}/(z - ib)$. Consider the rectangle:



The only pole of f in this rectangle is at ib and the residue is e^{-ab}, so it suffices to show that

$$\frac{1}{2\pi i} \left| \int_{R_T} f + \int_{L_T} f + \int_{\Gamma_T} f \right| \leq \frac{1}{Ta}(1 - e^{-Ta}) + e^{-Ta},$$

where R_T denotes the right vertical segment, L_T the left vertical segment and Γ_T the top vertical segment (all with the orientation given on the picture). We begin with

$$\frac{1}{2\pi i} \int_{R_T} f = \frac{1}{2\pi i} \int_0^T \frac{e^{ia(T+it)}}{T + it - ib} i dt.$$

Putting absolute values we get

$$\left| \frac{1}{2\pi i} \int_{R_T} f \right| \leq \frac{1}{2\pi T} \int_0^T e^{-at} dt = \frac{1}{2\pi Ta}(1 - e^{-aT}).$$

The same estimate holds for the left hand side, namely

$$\left| \frac{1}{2\pi i} \int_{L_T} f \right| \leq \frac{1}{2\pi Ta}(1 - e^{-aT}).$$

We now estimate the integral on the top segment. With the parametrization $t + iT$, $-T \leq t \leq T$ we get

$$\left|\frac{1}{2\pi i}\int_{\Gamma_T} f\right| \leq \frac{e^{-aT}}{2\pi}\int_{-T}^{T}\frac{dt}{|t+iT-ib|}$$
$$\leq \frac{e^{-aT}}{2\pi}\frac{2T}{T-b}.$$

Since $T > 2b$, we must have $2T/(T-b) \leq 4$ so that

$$\left|\frac{1}{2\pi i}\int_{\Gamma_T} f\right| \leq \frac{2e^{-aT}}{\pi}.$$

We see now that our estimate is sharper than the one we wanted to prove.

If a is negative, then a similar argument with a rectangle lying in the lower half plane gives

$$\left|\frac{1}{2\pi i}\int_{-T}^{T}\frac{e^{iaz}}{z-ib}dz - e^{-ba}\right| \leq \frac{1}{Ta}(e^{aT}-1)+e^{Ta}.$$

Exercise VI.2.7. *Let $c > 0$ and $a > 0$. Taking the integral over the vertical line, prove that*

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{a^z}{z}dz = \begin{cases} 0 & \text{if } a < 1, \\ \frac{1}{2} & \text{if } a = 1, \\ 1 & \text{if } a > 1. \end{cases}$$

If $a = 1$, the integral is to be interpreted as the limit

$$\int_{c-i\infty}^{c+i\infty} = \lim_{T\to\infty}\int_{c-iT}^{c+iT}.$$

[Hint: If $a > 1$, integrate around a rectangle with corners $c - Ai$, $c + Bi$, $-X + Bi$, $-X - Ai$, and let $X \to \infty$. If $a < 1$, replace $-x$ by x.]

Solution. Let $b = \log a$ so that

$$f(z) = \frac{a^z}{z} = \frac{e^{bz}}{z}.$$

We begin with the case $a = 1$. Then $b = 0$ and we must evaluate the integral

$$\int_{c-i\infty}^{c+i\infty}\frac{1}{z}dz.$$

If $X > 0$, the segment from $c - iX$ to $c + iX$ is parametrized by $c + it$ where $-X \leq t \leq X$, so that

$$\int_{c-i\infty}^{c+i\infty}\frac{1}{z}dz = \int_{-X}^{X}\frac{i}{c+it}dt.$$

Now

$$\int_{-X}^{X} \frac{i}{c+it}dt = i\int_{-X}^{X} \frac{c}{c^2+t^2}dt + \int_{-X}^{X} \frac{t}{c^2+t^2}dt$$
$$= 2i\arctan(X/c)$$

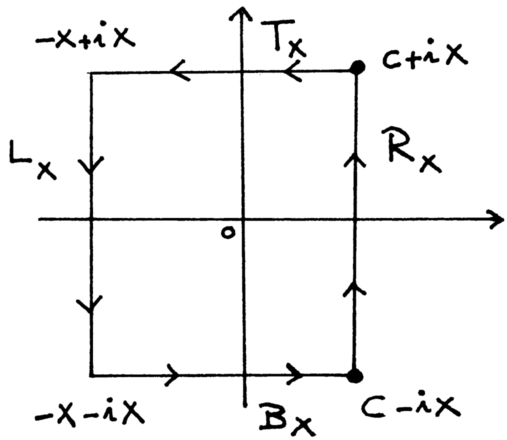
so letting $X \to \infty$ we obtain

$$\int_{c-i\infty}^{c+i\infty} \frac{1}{z}dz = 2i\frac{\pi}{2} = i\pi$$

and this proves that

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{a^z}{z}dz = \frac{1}{2}.$$

We now look at the case $a > 1$ or equivalently $b > 0$. Suppose $X > 0$ is large, and consider the contour:



Here, T_X denotes the horizontal segment on top, B_X the horizontal segment on the bottom, L_X the vertical segment on the left and R_X the vertical segment on the right and all segments have the orientation given on the picture. If γ is the path defined by

$$\gamma = R_X + T_X + L_X + B_X$$

the residue formula gives

$$\frac{1}{2\pi i}\int_{\gamma} \frac{a^z}{z}dz = \sum \text{residues of } f \text{ in } \gamma.$$

The only pole of f is at the origin and since the numerator is equal to 1 at 0 we conclude that the right hand side of the above equality is equal to 1. Therefore, it suffices to show that the integral over T_X, L_X and B_X go to 0 as $X \to \infty$. We begin with T_X. This segment is parametrized by $t + iX$ with $-X \leq t \leq c$ so that

$$\int_{T_X} \frac{a^z}{z} dz = \int_c^{-X} \frac{e^{b(t+iX)}}{t + iX} dt,$$

and therefore

$$\begin{aligned} \left| \int_{T_X} \frac{a^z}{z} dz \right| &\leq \int_{-X}^{c} \frac{e^{bt}}{|t + iX|} dt \\ &\leq \frac{1}{X} \int_{-X}^{c} e^{bt} dt = \frac{1}{Xb} \left[e^{bc} - e^{-bX} \right], \end{aligned}$$

which implies that

$$\left| \int_{T_X} \frac{a^z}{z} dz \right| \to 0 \quad \text{as } X \to \infty.$$

For L_X, we use the parametrization $-X + it$ where $-X \leq t \leq X$ so that

$$\int_{L_X} \frac{a^z}{z} dz = \int_{-X}^{X} i \frac{e^{b(-X+it)}}{-X + it} dt.$$

Therefore

$$\begin{aligned} \left| \int_{L_X} \frac{a^z}{z} dz \right| &\leq \int_{-X}^{X} \frac{e^{-bX}}{|t + iX|} dt \\ &\leq \frac{e^{-bX}}{X} \int_{-X}^{X} dt \leq 2e^{-bX}, \end{aligned}$$

and this proves that

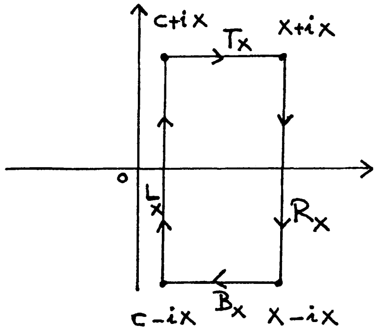
$$\left| \int_{L_X} \frac{a^z}{z} dz \right| \to 0 \quad \text{as } X \to \infty.$$

Finally, we must show that the integral over B_X tends to 0 as $X \to \infty$. To do this, we use the parametrization $t - iX$ where $-X \leq t \leq c$, and estimating as before one easily finds that

$$\left| \int_{B_X} \frac{a^z}{z} dz \right| \leq \frac{1}{Xb} \left[e^{bc} - e^{-bX} \right],$$

and this settles the case $a > 1$.

For the case $a < 1$ or equivalently $b < 0$ we consider the following contour:



If $\gamma = R_X + T_X + L_X + B_X$, then the residue formula gives

$$\frac{1}{2\pi i}\int_\gamma \frac{a^z}{z}dz = \sum \text{residues of } f \text{ in } \gamma.$$

so it suffices to show that the integral over R_X, T_X and B_X tend to 0 as $X \to \infty$. To prove this, we argue as before. With the obvious parametrizations we obtain

$$\left|\int_{T_X} \frac{a^z}{z}dz\right| \leq \frac{1}{bX}\left[e^{bX} - e^{bc}\right],$$

and the right hand side goes to 0 as $X \to \infty$. Similarly, we obtain that

$$\left|\int_{B_X} \frac{a^z}{z}dz\right| \to 0 \quad \text{and} \quad \left|\int_{R_X} \frac{a^z}{z}dz\right| \to 0$$

as $X \to 0$ and this concludes the proof.

Exercise VI.2.8. *(a) Show that for $a > 0$ we have*

$$\int_{-\infty}^{\infty} \frac{\cos x}{(x^2+a^2)^2}dx = \frac{\pi(1+a)}{2a^3e^a}.$$

(b) Show that for $a > b > 0$ we have

$$\int_0^{\infty} \frac{\cos x}{(x^2+a^2)(x^2+b^2)}dx = \frac{\pi}{a^2-b^2}\left(\frac{1}{be^b} - \frac{1}{ae^a}\right).$$

Solution. The function $\sin x$ is odd so $\int_{-\infty}^{\infty} \sin x/(x^2+a^2)^2 dx = 0$ and therefore

$$\int_{-\infty}^{\infty} \frac{\cos x}{(x^2+a^2)^2}dx = \int_{-\infty}^{\infty} \frac{e^{ix}}{(x^2+a^2)^2}dx.$$

Let $f(z) = 1/(z^2+a^2)^2$. We want to find the Fourier transform $\int_{-\infty}^{\infty} f(x)e^{ix}dx$. An estimate like in Exercise 5 shows that we can apply Theorem 2.2, and therefore

$$\int_{-\infty}^{\infty} f(x)e^{ix}dx = 2\pi i \sum \text{residues of } f(z)e^{iz} \text{ in the upper half plane.}$$

The only pole of f in the upper half plane is at ia. We must now find the residue of f at this pole. We write

$$f(z) = \frac{1}{(z-ia)^2(z+ia)^2}.$$

Now we have

$$(z+ia)^{-2} = (z-ia+2ia)^{-2} = (2ia)^{-2}\left(1+\frac{z-ia}{2ia}\right)^{-2}$$

which after expanding becomes

$$(z+ia)^{-2} = (2ia)^{-2}\left(1-2\frac{z-ia}{2ia}+\cdots\right).$$

We also have $e^{iz} = e^{-a}e^{i(z-ia)} = e^{-a}(1+i(z-ia)+\cdots)$ so

$$f(z)e^{iz} = \frac{e^{-a}}{(z-ia)^2(2ia)^2}\left(1-2\frac{z-ia}{2ia}+\cdots\right)(1+i(z-ia)+\cdots).$$

Hence

$$\operatorname{res}_{ia} f(z)e^{iz} = \frac{e^{-a}}{(2ia)^2}\left(\frac{-1}{ia}+i\right) = \frac{e^{-a}(1+a)}{4a^3 i}.$$

By the residue formula we conclude that

$$\int_{-\infty}^{\infty} f(x)e^{ix}dx = 2\pi i\frac{e^{-a}(1+a)}{4a^3 i} = \pi\frac{e^{-a}(1+a)}{2a^3}$$

as was to be shown.

(b) Arguing like in (a) and using the fact that cos is even we find that the desired integral is equal to $\frac{1}{2}\int_{-\infty}^{\infty} f(x)e^{ix}dx$ where

$$f(z) = \frac{1}{(z^2+a^2)(z^2+b^2)}.$$

We can apply Theorem 2.2. We are only concerned with singularities in the upper half plane. In this region f has two simple poles one at ia and the other at ib. Computing the derivative of (z^2+a^2) implies that the residue of $f(z)e^{iz}$ at ia is

$$\operatorname{res}_{z=ia} f(z)e^{iz} = \frac{e^{i(ia)}}{(2ia)((ia)^2+b^2)} = -\frac{e^{-a}}{(2ia)(a^2-b^2)}.$$

Similarly we find that

$$\operatorname{res}_{z=ib} f(z)e^{iz} = \frac{e^{i(ib)}}{(a^2+(ib)^2)(2ia)} = -\frac{e^{-b}}{(2ib)(a^2-b^2)}.$$

By Theorem 2.2 we obtain

$$\begin{aligned}\int_{-\infty}^{\infty} f(x)e^{ix}dx &= 2\pi i(\operatorname{res}_{z=ia} f(z)e^{iz} + \operatorname{res}_{z=ib} f(z)e^{iz}) \\ &= \frac{\pi}{a^2-b^2}\left(-\frac{e^{-a}}{a}+\frac{e^{-b}}{b}\right).\end{aligned}$$

Conclude.

Exercise VI.2.9. $\int_0^\infty \frac{\sin^2 x}{x^2}dx = \pi/2$. *[Hint: Consider the integral of* $(1 - e^{2ix})/x^2$.*]*

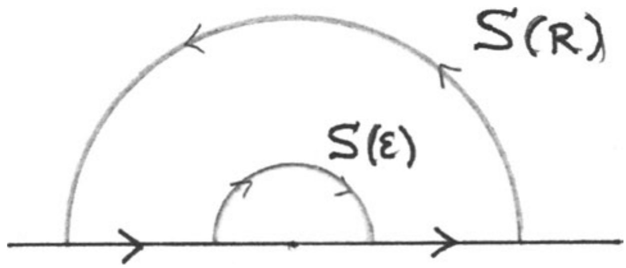
Solution. Since the integrand is even, the desired integral is equal to

$$\frac{1}{2}\int_{-\infty}^{\infty} \frac{\sin^2 x}{x^2}dx.$$

The trigonometric identity $2\sin^2 x = 1 - \cos 2x$, implies

$$2\int_{-\infty}^{\infty} \frac{\sin^2 x}{x^2}dx = \mathrm{Re}\left(\int_{-\infty}^{\infty} \frac{1-e^{2ix}}{x^2}dx\right).$$

We have reduced the problem to finding the integral $\int_{-\infty}^{\infty} f(x)dx$ where $f(z) = (1 - e^{2iz})/z^2$. The function f has a unique pole at the origin. We take as a path



To show that

$$\lim_{R\to\infty} \int_{S(R)} f(z)dz = 0$$

split the integral and write is as

$$\int_{S(R)} \frac{dz}{z^2} - \int_{S(R)} \frac{e^{2iz}}{z^2}dz.$$

The first integral goes to 0 as R tends to infinity because it is bounded by $\pi R/R^2$, namely the sup norm of $1/z^2$ on $S(R)$ times the length of $S(R)$. The second integral is estimated exactly like on page 196 of Lang's book. By the lemma on this same page we obtain

$$\lim_{\epsilon\to 0} \int_{S(\epsilon)} f(z)dz = -\pi i \,\mathrm{res}_{z=0} f(z).$$

To find the residue, we must use the power series expansion of the exponential

$$f(z) = \frac{1-(1+2iz+(2iz)^2/2!+\cdots)}{z^2} = \frac{-2i}{z} + \text{terms of higher order.}$$

Hence the residue of f at the origin is $-2i$ and therefore

$$\int_{-\infty}^{\infty} \frac{1-e^{2ix}}{x^2} dx = 2\pi.$$

Conclude.

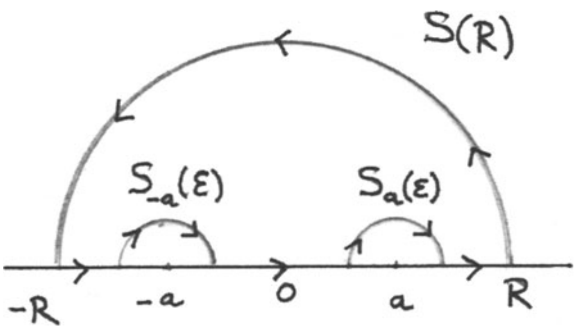
Exercise VI.2.10. $\int_{-\infty}^{\infty} \frac{\cos x}{a^2-x^2} dx = \frac{\pi \sin a}{a}$ *for* $a > 0$. *The integral is meant to be interpreted as the limit:*

$$\lim_{B\to\infty} \lim_{\delta\to 0} \int_{-B}^{-a-\delta} + \int_{-a+\delta}^{a-\delta} + \int_{a+\delta}^{B}.$$

Solution. Since the sine function is odd, the integral we must compute is equal to

$$\int_{-\infty}^{\infty} f(x)dx \quad \text{where } f(z) = \frac{e^{iz}}{a^2 - z^2}.$$

The function f has two simple poles, one at a and the other at $-a$. Consider the following contour:



We must show that

$$\lim_{R\to\infty} \int_{S(R)} f(z)dz = 0.$$

We argue like on page 196 of Lang's book. We have

$$\int_{S(R)} f(z)dz = \int_0^{\pi} \frac{e^{iR\cos\theta} e^{-R\sin\theta}}{a^2 - R^2 e^{2i\theta}} iRe^{i\theta} d\theta,$$

so for all large R we get

$$\left| \int_{S(R)} f(z)dz \right| \leq \int_0^{\pi} \frac{e^{-R\sin\theta}}{R^2 - a^2} R d\theta = \frac{2R}{R^2 - a^2} \int_0^{\pi/2} e^{-R\sin\theta} d\theta.$$

But if $0 \le \theta \le \pi/2$, then $\sin\theta \ge 2\theta/\pi$, thus

$$\left|\int_{S(R)} f(z)dz\right| \le \frac{2R}{R^2-a^2}\int_0^{\pi/2} e^{-2R\theta/\pi}d\theta = \frac{\pi}{R^2-a^2}(1-e^{-R}),$$

and now it is clear that our limit holds.

Now we must evaluate the limits

$$\lim_{\epsilon\to 0}\int_{S_a(\epsilon)} f(z)dz \quad\text{and}\quad \lim_{\epsilon\to 0}\int_{S_{-a}(\epsilon)} f(z)dz.$$

A simple modification of the lemma on page 196 of Lang's book shows that if f has a pole at x, then

$$\lim_{\epsilon\to 0}\int_{S_x(\epsilon)} f(z)dz = \pi \operatorname{res}_{z=x} f(z).$$

Writing f as

$$f(z) = \frac{e^{iz}}{(a-z)(a+z)}$$

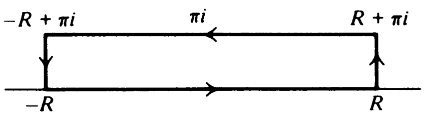
we find that

$$\operatorname{res}_{z=a} f(z) = \frac{-e^{ia}}{2a} \quad\text{and}\quad \operatorname{res}_{z=-a} f(z) = \frac{e^{-ia}}{2a}.$$

Therefore

$$\begin{aligned}\int_{-\infty}^{\infty} f(z)dz &= \pi\left(\frac{-e^{ia}}{2a}+\frac{e^{-ia}}{2a}\right)\\ &= \frac{\pi}{a}\left(\frac{e^{ia}}{2i}-\frac{e^{-ia}}{2i}\right)\\ &= \frac{\pi\sin a}{a}.\end{aligned}$$

Exercise VI.2.11. $\int_{-\infty}^{\infty}\frac{\cos x}{e^x+e^{-x}}dx = \frac{\pi}{e^{\pi/2}+e^{-\pi/2}}$. *Use the indicated contour:*



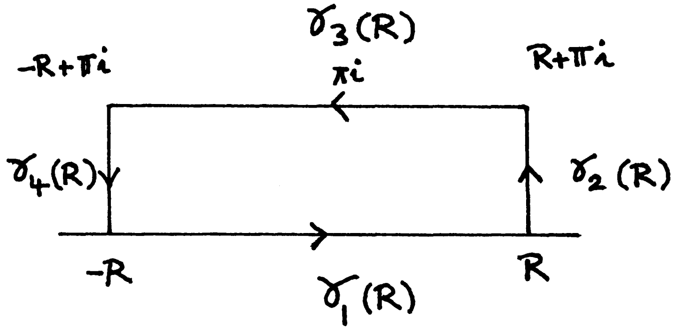
Solution. The sine function is odd, so the desired integral is equal to

$$\int_{-\infty}^{\infty} f(x)dx \quad\text{where } f(z) = \frac{e^{iz}}{e^z+e^{-z}}.$$

To find the singularities of f we must solve $e^z+e^{-z}=0$. Multiplying this equation by e^z we get $e^{2z}+1=0$. Letting $z=x+iy$, we get $e^{2x}e^{2iy}=-1$.

Putting absolute values we find $x = 0$ and this shows that f has singularities at the points $i(\pi/2 + k\pi)$ where $k \in \mathbf{Z}$.

Consider the contour $\gamma(R) = \gamma_1(R) + \gamma_2(R) + \gamma_3(R) + \gamma_4(R)$ as shown on the figure



The only singularity of f in the interior of the contour is at $i\pi/2$. The derivative of $e^z + e^{-z}$ at that point is equal to $2i$ which is nonzero so f has a simple pole at $i\pi/2$ with

$$\text{res}_{z=i\pi/2} f(z) = \frac{e^{i(i\pi/2)}}{2i} = \frac{e^{-\pi/2}}{2i}.$$

By the residue formula, we get

$$\int_{\gamma(R)} f(z)dz = \pi e^{-\pi/2}.$$

We now want show that the integral over $\gamma_2(R)$ and $\gamma_4(R)$ tend to 0 as R tends to infinity. We can estimate the integral by

$$\left|\int_{\gamma_2(R)} f(z)dz\right| \leq \int_{\gamma_2(R)} |f(z)|\, d \leq \pi \sup_{0\leq y\leq \pi} \left|\frac{e^{iR}e^{-y}}{e^R e^{iy} + e^{-R}e^{-iy}}\right|,$$

and for large R

$$\left|\frac{e^{iR}e^{-y}}{e^R e^{iy} + e^{-R}e^{-iy}}\right| \leq \frac{e^{-y}}{e^R\left|e^{iy} + e^{-2R}e^{-iy}\right|} \leq \frac{1}{e^R(1 - e^{-2R})}.$$

The last inequality follows from $0 \leq y \leq \pi$ and the triangle inequality applied to the denominator and the fact that R is large. It is now clear that the integral of f over $\gamma_2(R)$ tends to 0 as R tends to infinity. A similar argument proves the same result for the integral of f over $\gamma_4(R)$.

Finally, we find the expression of the integral of f over $\gamma_3(R)$. Using the parametrization $t + \pi$ for $-R \leq t \leq R$ and being careful about the orientation we get

$$\begin{aligned}\int_{\gamma_3(R)} f(z)dz &= \int_R^{-R} \frac{e^{it+\pi}}{e^{t+\pi i}+e^{-t\pi i}}dt\\ &= e^{-\pi}\int_R^{-R}\frac{e^{it}}{-e^t-e^{-t}}dt\\ &= e^{-\pi}\int_{-R}^{R}\frac{e^{it}}{e^t+e^{-t}}dt\\ &= e^{-\pi}\int_{\gamma_1(R)} f(z)dz.\end{aligned}$$

So if I denotes the integral we want to evaluate we conclude that

$$I + e^{-\pi} I = \pi e^{-\pi/2},$$

and therefore

$$I = \frac{\pi}{e^{\pi/2}+e^{-\pi/2}}.$$

This concludes the exercise.

Exercise VI.2.12. $\int_0^\infty \frac{x\sin x}{x^2+a^2}dx = \frac{1}{2}\pi e^{-a}$ *if* $a > 0$.

Solution. The integral we wish to evaluate has an even integrand so it is equal to

$$\frac{1}{2}\int_{-\infty}^{\infty}\frac{x\sin x}{x^2+a^2}dx.$$

The function $x\cos x$ is odd so

$$\int_{-\infty}^{\infty}\frac{x\sin x}{x^2+a^2}dx = \operatorname{Im}\left(\int_{-\infty}^{\infty} f(x)e^{ix}dx\right) \quad \text{where } f(z) = \frac{z}{z^2+a^2}.$$

Clearly, the function f verifies the hypothesis of Theorem 2.2 so we can apply the formula

$$\int_{-\infty}^{\infty} f(x)e^{ix}dx = 2\pi i \sum \text{residues of } f(z)e^{iz} \text{ in the upper half plane.}$$

The function f has simple poles at ia and $-ia$. Since $a > 0$ we are only concerned with the pole at ia which is in the upper half plane. Since

$$f(z) = \frac{z}{(z-ia)(z+ia)},$$

it follows that

$$\operatorname{res}_{z=ia} f(z)e^{iz} = \left(\frac{ia}{2ia}\right)e^{i(ia)} = \frac{e^{-a}}{2}.$$

Hence

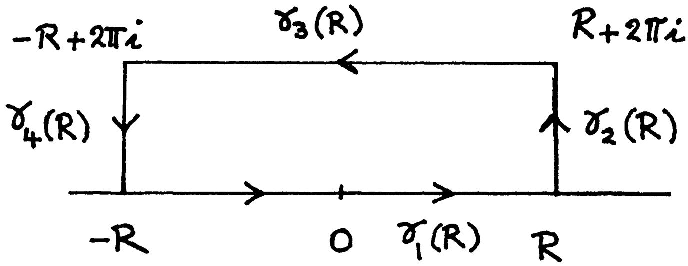
$$\int_{-\infty}^{\infty} f(x)e^{ix}dx = \pi i e^{-a}.$$

The observations at the beginning of the exercise imply that

$$\int_0^{\infty} \frac{x \sin x}{x^2 + a^2} dx = \frac{1}{2}\pi e^{-a}.$$

Exercise VI.2.13. $\int_{-\infty}^{\infty} \frac{e^{ax}}{e^x+1} dx = \frac{\pi}{\sin \pi a}$ *for* $0 < a < 1$.

Solution. The solution to this exercise is very much like our answer to Exercise VI.2.11. Let $f(z) = e^{az}/(e^z + 1)$. The function f has poles at $i\pi + 2k\pi$ with $k \in \mathbf{Z}$. Consider the contour $\gamma(R) = \gamma_1(R) + \gamma_2(R) + \gamma_3(R) + \gamma_4(R)$ given by



Taking the derivative of the denominator of f we find that the residue of f at $i\pi$ is $e^{ai\pi}/e^{i\pi} = -e^{ai\pi}$ so by the residue formula we obtain

$$\int_{\gamma(R)} f(z)dz = -2\pi i e^{ai\pi}.$$

We must show that the integrals on the sides $\gamma_2(R)$ and $\gamma_4(R)$ tend to 0 as R tends to infinity. We estimate the sup norm of f on $\gamma_2(R)$ by

$$\sup_{z \in \gamma_2(R)} |f(z)| = \sup_{z \in \gamma_2(R)} \left| \frac{e^{aR}e^{iay}}{e^R e^{iy} + 1} \right| \leq \frac{e^{aR}}{e^R - 1}.$$

But $0 < a < 1$ so we see that the sup norm of f on $\gamma_2(R)$ goes to 0 as R tends to infinity, and since $\gamma_2(R)$ has length 2π we conclude that the integral of f over $\gamma_2(R)$ tends to 0 as R tends to infinity. A similar argument shows that the same conclusion holds for the integral of f over $\gamma_4(R)$.

We must now find an expression for the integral of f over $\gamma_3(R)$. Arguing like in Exercise 11 we find that

$$\int_{\gamma_3(R)} f(z)dz = -e^{2\pi ai} \int_{\gamma_1(R)} f(z)dz.$$

If I denotes the integral we want to compute, we get (letting $R \to \infty$)

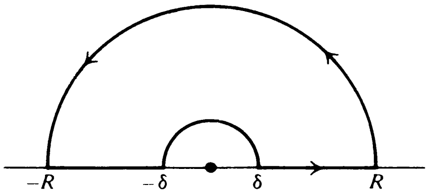
$$I - e^{2\pi ai} I = -2\pi i e^{ai\pi}$$

so that

$$\frac{(e^{\pi ai} - e^{-\pi ai})}{2i} I = \pi.$$

We have therefore proved that $I = \pi/(\sin \pi a)$.

Exercise VI.2.14. *(a)* $\int_0^\infty \frac{(\log x)^2}{1+x^2} dx = \pi^3/8$. *Use the contour*

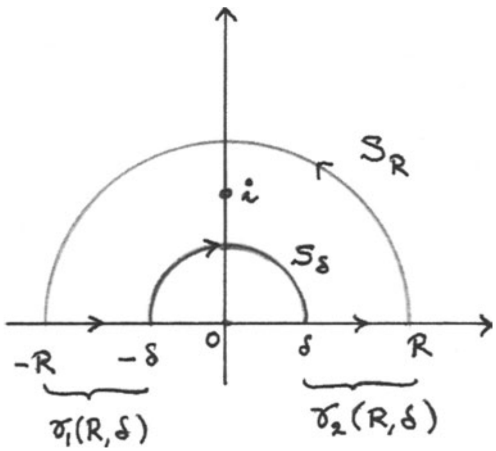


(b) $\int_0^\infty \frac{\log x}{(x^2+1)^2} dx = -\pi/4$.

Solution. (a) We first define the following mysterious function:

$$f(z) = \frac{(\log z - \frac{i\pi}{2})^2}{1 + z^2}.$$

We take the branch of the logarithm given by deleting the negative imaginary axis and taking the angle to be $-\pi/2 < \theta < 3\pi/2$. Consider the contour given by



The only singularity of f which is of interest is the simple pole at i. The residue of f at that pole is

$$\frac{(\log i - i\pi/2)^2}{2i} = 0.$$

This is one reason which explains the strange constant $\pi i/2$ in the definition of f. By the residue formula, we conclude that $\int_\gamma f(z)dz = 0$. The integral of f on S_R tends to 0 as $R \to \infty$ because the length of S_R multiplied by the sup norm on S_R behaves like $R\frac{(\log R)^2}{R^2}$ which tends to 0 as R tends to infinity. The integral of f on S_δ behaves like $(\log\delta)^2\delta$ which tends to 0 as $\delta \to 0$.

On the real axis we have

$$\int_{\gamma_1(R,\delta)} f(x)dx = \int_{-R}^{-\delta} \frac{(\log|x| + i(\pi/2))^2}{1+x^2}dx$$

and

$$\int_{\gamma_2(R,\delta)} f(x)dx = \int_{\delta}^{R} \frac{(\log|x| - i(\pi/2))^2}{1+x^2}dx.$$

Letting $R \to \infty$ and $\delta \to 0$ we see that after cancellations (which explain the choice of our f) we get

$$\int_{-\infty}^{0} \frac{(\log|x|)^2}{1+x^2}dx + \int_{0}^{\infty} \frac{(\log|x|)^2}{1+x^2}dx - \frac{\pi^2}{4}\int_{-\infty}^{\infty}\frac{dx}{1+x^2} = 0,$$

hence

$$2\int_0^\infty \frac{(\log x)^2}{1+x^2}dx = \frac{\pi^2}{4}\int_{-\infty}^{\infty}\frac{dx}{1+x^2} = \frac{\pi^3}{4}.$$

(b) We use the same technique as in (a). Let

$$f(z) = \frac{\log z - \frac{i\pi}{2}}{(z^2+1)^2}.$$

We use the same branch of the logarithm and the same contour as in part (a). The only singularity of f in the upper half plane is at the point i. Our next step is to find the residue of f at this singularity. Since we can write

$$f(z) = \frac{\log z - \frac{i\pi}{2}}{(z+i)^2(z-i)^2}$$

it suffices to find the coefficient of the term $z - i$ in the power series expansion of $(\log z - i\pi/2)/(z+i)^2$ near i. We simply have

$$\frac{1}{(z+i)^2} = \frac{1}{(2i)^2\left(1+\frac{z-i}{2i}\right)^2} = \frac{-1}{4}\left(1 - 2\frac{z-i}{2i} + \text{higher order terms}\right),$$

and

$$\log z - i\pi/2 = \sum \frac{(-1)^{n-1}}{n}\left(\frac{z-i}{i}\right)^n = \frac{z-i}{i} + \text{higher order terms}.$$

Thus

$$\operatorname{res}_{z=i} f(z) = \frac{-1}{4i}.$$

The residue formula gives

$$\int_\gamma f(z)dz = 2\pi i \operatorname{res}_{z=i} f(z) = \frac{-\pi}{2}.$$

An argument similar to the one given in (a) shows that the integrals on the semicircles S_R and S_δ tend to 0 as $R \to \infty$ and $\delta \to 0$ respectively. Therefore

$$\int_{-\infty}^0 \frac{\log|x| + i\pi/2}{(x^2+1)^2}dx + \int_0^\infty \frac{\log|x| - i\pi/2}{(x^2+1)^2}dx = \frac{-\pi}{2}.$$

We obtain

$$2\int_0^\infty \frac{\log x}{(x^2+1)^2}dx = \frac{-\pi}{2},$$

as was to be shown.

Exercise VI.2.15. *(a)* $\int_0^\infty \frac{x^a}{1+x}\frac{dx}{x} = \frac{\pi}{\sin \pi a}$ *for* $0 < a < 1$.
(b) $\int_0^\infty \frac{x^a}{1+x^3}\frac{dx}{x} = \frac{\pi}{3\sin(\pi a/3)}$ *for* $0 < a < 3$.

Solution. Let $f(z) = 1/(1+z)$. Then $|f(z)| \leq C/|z|$ as $|z| \to \infty$ for some constant C and $|f(z)| \to 1$ as $|z| \to 0$, so we can apply Theorem 2.4 which states that the integral (a Mellin transform)

$$\int_0^\infty f(x)x^a\frac{dx}{x}$$

is equal to $-\frac{\pi e^{-\pi i a}}{\sin \pi a}$ times the sum of the residues of $f(z)z^{a-1}$ at the poles of f, excluding the residue at 0.

The only pole of f is at -1 and

$$\operatorname{res}_{z=-1} f(z)z^{a-1} = (-1)^{a-1} = e^{(a-1)\log(-1)} = e^{(a-1)i\pi}.$$

Therefore

$$\int_0^\infty \frac{x^a}{1+x}\frac{dx}{x} = -\frac{\pi e^{-\pi i a}}{\sin \pi a}e^{(a-1)i\pi} = \frac{\pi}{\sin \pi a}.$$

(b) As in part (a), we can apply Theorem 2.4, so all we have to do is compute the residues of $f(z)z^{a-1}$ where $f(z) = 1/(1+z^3)$. The poles of f are at $e^{i\pi/3}$, $e^{i\pi}$ and $e^{5i\pi/3}$ so the sum of the residues of $f(z)z^{a-1}$ excluding the residue at the origin is

$$\frac{(e^{i\pi/3})^{a-1}}{3(e^{i\pi/3})^2} + \frac{(e^{i\pi})^{a-1}}{3(e^{i\pi})^2} + \frac{(e^{5i\pi/3})^{a-1}}{3(e^{5i\pi/3})^2}.$$

We transform the first term in the following way

$$\frac{(e^{i\pi/3})^{a-1}}{3(e^{i\pi/3})^2} = e^{(a-1)(i\pi/3)}3e^{2i\pi/3} = \frac{e^{ai\pi/3}e^{-i\pi}}{3} = -\frac{e^{ai\pi/3}}{3}.$$

Making the same transformations to the other terms, we find that the sum of the residues of $f(z)z^{a-1}$ excluding the residue at the origin is

$$= \frac{-1}{3}\left(e^{ai\pi/3} + e^{ai\pi} + e^{ai5\pi/3}\right)$$
$$= \frac{-e^{ai\pi}}{3}\left(e^{ai(-2)\pi/3} + 1 + e^{ai2\pi/3}\right).$$

Hence

$$\int_0^\infty \frac{x^a}{1+x^3}\frac{dx}{x} = \frac{\pi}{3\sin\pi a}\left(e^{ai(-2)\pi/3} + 1 + e^{ai2\pi/3}\right).$$

We claim that

$$\frac{e^{ai(-2)\pi/3} + 1 + e^{ai2\pi/3}}{\sin\pi a} = \frac{1}{\sin(\pi a/3)}.$$

Using Euler's formula $2i\sin\theta = e^{i\theta} - e^{-i\theta}$ to write everything with exponentials and cross multiplying proves our claim.

Exercise VI.2.16. *Let f be a continuous function, and suppose that the integral*

$$\int_0^\infty f(x)x^a\frac{dx}{x}$$

is absolutely convergent. Show that it is equal to the integral

$$\int_{-\infty}^\infty f(e^t)e^{at}dt.$$

If we put $g(t) = f(e^t)$, this shows that the Mellin transform is essentially a Fourier transform, up to a change of variable.

Solution. We change variables $e^t = x$. Then $dx = e^t dt$ and therefore

$$\int_0^\infty f(x)x^a\frac{dx}{x} = \int_{-\infty}^\infty f(e^t)(e^t)^a e^t\frac{dt}{e^t} = \int_{-\infty}^\infty f(e^t)e^{at}dt.$$

Exercise VI.2.17. $\int_0^{2\pi}\frac{1}{1+a^2-2a\cos\theta}d\theta = \frac{2\pi}{1-a^2}$ *if $0 < a < 1$. The answer comes out to the negative of that if $a > 1$.*

Solution. Since this is a trigonometric integral we will apply Theorem 2.3. We have

$$f(z) = \frac{1}{iz}\frac{1}{1+a^2-2a\left(\frac{1}{2}\left(z+\frac{1}{z}\right)\right)} = \frac{1}{i}\frac{1}{-az^2+(1+a^2)z-a}.$$

The roots of the denominator of the second fraction are

$$z_1 = \frac{-(1+a^2)+\sqrt{(1-a^2)^2}}{-2a} \quad\text{and}\quad z_2 = \frac{-(1+a^2)-\sqrt{(1-a^2)^2}}{-2a}.$$

If $0 < a < 1$, the only pole of f in the unit circle is at $z_1 = a$ and (differentiating the denominator of the fraction) we find that the residue is

$$\frac{1}{i}\frac{1}{-2az^1 + (1+a^2)} = \frac{1}{i(1-a^2)},$$

and therefore

$$\int_C f(z)dz = 2\pi i\left(\frac{1}{i(1-a^2)}\right) = \frac{2\pi}{1-a^2}.$$

If $a > 1$ the only pole of f in the unit circle is at $z_1 = 1/a$ and the residue is

$$\frac{1}{i}\frac{1}{-2az^1 + (1+a^2)} = \frac{1}{i(-1+a^2)},$$

hence

$$\int_C f(z)dz = \frac{2\pi}{a^2-1}.$$

Exercise VI.2.18. $\int_0^\pi \frac{1}{1+\sin^2\theta}d\theta = \frac{\pi}{\sqrt{2}}$.

Solution. See Exercise 20.

Exercise VI.2.19. $\int_0^\pi \frac{1}{3+2\cos\theta}d\theta = \frac{\pi}{\sqrt{5}}$.

Solution. In order to apply Theorem 2.3 we must integrate from 0 to 2π. We claim that

$$\int_0^\pi \frac{1}{3+2\cos\theta}d\theta = \frac{1}{2}\int_0^{2\pi}\frac{1}{3+2\cos\theta}d\theta.$$

To prove this claim, we change variables $\theta \to -\theta$ in the first integral so that

$$\int_0^\pi \frac{1}{3+2\cos\theta}d\theta = \int_0^{-\pi}\frac{-1}{3+2\cos(-\theta)}d\theta = \int_{-\pi}^0 \frac{1}{3+2\cos\theta}d\theta.$$

Now changing variables $\theta \to \theta + 2\pi$ we get

$$\int_{-\pi}^0 \frac{1}{3+2\cos\theta}d\theta = \int_\pi^{2\pi}\frac{1}{3+2\cos\theta}d\theta.$$

This proves our claim. We must now compute

$$\int_0^{2\pi}\frac{1}{3+2\cos\theta}d\theta$$

and we use Theorem 2.3 with the function

$$f(z) = \frac{1}{iz}\frac{1}{3+2\frac{1}{2}\left(z+\frac{1}{z}\right)} = \frac{1}{i(z^2+3z+1)}.$$

The zeros of the denominator are

$$z_1 = \frac{-3+\sqrt{5}}{2} \quad \text{and} \quad z_2 = \frac{-3-\sqrt{5}}{2}.$$

The only pole of f in the unit circle is at z_1 and the residue is

$$\frac{1}{i(2z_1+3)} = \frac{1}{i\sqrt{5}},$$

and therefore

$$\int_0^{2\pi} \frac{d\theta}{3+2\cos\theta} = 2\pi i \frac{1}{2i\sqrt{5}} = \frac{2\pi}{\sqrt{5}}.$$

This proves that

$$\int_0^{\pi} \frac{d\theta}{3+2\cos\theta} = \frac{\pi}{\sqrt{5}}.$$

Exercise VI.2.20. $\int_0^{\pi} \frac{ad\theta}{a^2+\sin^2\theta} = \int_0^{2\pi} \frac{ad\theta}{1+2a^2-\cos\theta} = \frac{\pi}{\sqrt{1+a^2}}.$

Solution. We have

$$a^2+\sin^2\theta = a^2 + \frac{1-\cos 2\theta}{2} = \frac{1}{2}\left(2a^2+1-\cos 2\theta\right),$$

so changing variables $\varphi = 2\theta$ we find that

$$\int_0^{\pi} \frac{ad\theta}{a^2+\sin^2\theta} = \int_0^{2\pi} \frac{ad\theta}{1+2a^2-\cos\theta} = \frac{\pi}{\sqrt{1+a^2}}.$$

To compute this last integral, we use Theorem 2.3 with

$$f(z) = \frac{1}{iz}\frac{a}{1+2a^2-\left(\frac{1}{2}\left(z+\frac{1}{z}\right)\right)} = \frac{2ai}{z^2-(2+4a^2)z+1}.$$

The roots of the denominator are

$$z_1 = \frac{2+4a^2+\sqrt{16a^2+16a^4}}{2} = 1+2a^2+2|a|\sqrt{1+a^2},$$

and

$$z_2 = 1+2a^2-2|a|\sqrt{1+a^2}.$$

The only pole of f in the unit circle is at z_2 and the residue of f at this point is

$$\frac{2ai}{2z_2-(2+4a^2)} = \frac{ai}{-2|a|\sqrt{1+a^2}}$$

and therefore

$$\int_C f(z)dz = 2\pi i \frac{ai}{-2|a|\sqrt{1+a^2}} = \frac{a}{|a|}\frac{\pi}{\sqrt{1+a^2}}.$$

Conclude.

Exercise VI.2.21. $\int_0^{\pi/2} \frac{1}{(a+\sin^2\theta)^2} d\theta = \frac{\pi(2a+1)}{4(a^2+a)^{3/2}}$ *for* $a > 0$.

Solution. Using the fact that

$$\sin^2\theta = \frac{1}{2}(1-\cos 2\theta)$$

and arguing like at the beginning of Exercise 19, one finds after a few linear changes of variables that

$$\int_0^{\pi/2} \frac{1}{(a+\sin^2\theta)^2} d\theta = \int_0^{2\pi} \frac{d\theta}{(2a+1-\cos\theta)^2}.$$

Since we reduced the problem to a trigonometric integral from 0 to 2π we can apply Theorem 2.3 with the function.

$$\begin{aligned} f(z) &= \frac{1}{iz} \frac{1}{\left(2a+1-\frac{1}{2}\left(z+\frac{1}{z}\right)\right)^2} \\ &= \frac{z}{i\left(-\frac{z^2}{2}+(2a+1)z-\frac{1}{2}\right)^2}. \end{aligned}$$

The zeros of the denominator are at the points

$$z_1 = (2a+1) - 2\sqrt{a^2+a} \quad \text{and} \quad z_2 = (2a+1) + 2\sqrt{a^2+a}.$$

Since z_1 is the only pole of f in the unit circle we must compute the residue of f at this point. We write

$$f(z) = \frac{z}{i(1/4)(z-z_1)^2(z-z_2)^2} = \frac{4z}{i(z-z_1)^2(z-z_2)^2},$$

so that the residue of f is equal to the coefficient of $z - z_1$ in the power series expansion of

$$h(z) = \frac{4z}{i(z-z_2)^2}$$

near z_1. To find this coefficient, we first differentiate h and obtain

$$h'(z) = \frac{4}{i}\left[\frac{1}{(z-z_2)^2} - 2\frac{z}{(z-z_2)^3}\right] = \frac{4}{i}\left[\frac{-z-z^2}{(z-z_2)^3}\right],$$

which we evaluate at z_1 to obtain the residue of f at z_1

$$\operatorname{res}_{z=z_1} f(z) = h'(z_1) = \frac{4}{i}\frac{-4a-2}{-4^3(a^2+a)^{3/2}} = \frac{1}{8i}\frac{2a+1}{(a^2+a)^{3/2}}.$$

Therefore

$$\int_C f(z)dz = 2\pi i \frac{1}{8i}\frac{2a+1}{(a^2+a)^{3/2}} = \frac{\pi(2a+1)}{4(a^2+a)^{3/2}}.$$

Exercise VI.2.22. $\int_0^{2\pi} \frac{1}{2-\sin\theta} d\theta = 2\pi/\sqrt{3}$.

Solution. We will apply Theorem 2.3 with the function

$$f(z) = \frac{1}{iz}\frac{1}{2-\frac{1}{2i}\left(z-\frac{1}{z}\right)} = \frac{2}{-z^2+4iz+1}.$$

The roots of the denominator are

$$z_1 = 2i - i\sqrt{3} \quad \text{and} \quad z_2 = 2i + i\sqrt{3}.$$

The only pole of f in the unit circle is at z_1 and the residue of f at this point is

$$\frac{2}{-2z_1+4i}=\frac{1}{i\sqrt{3}}.$$

Hence

$$\int_C f(z)dz=2\pi i\frac{1}{i\sqrt{3}}=\frac{2\pi}{\sqrt{3}}.$$

Exercise VI.2.23. $\int_0^{2\pi}\frac{1}{(a+b\cos\theta)^2}d\theta=\frac{2\pi a}{(a^2-b^2)^{3/2}}$ *for* $0<b<a$.

Solution. We will apply Theorem 2.3 with

$$f(z)=\frac{1}{iz}\frac{1}{\left(a+\frac{b}{2}\left(z+\frac{1}{z}\right)\right)^2}=\frac{z}{i\left(\frac{b}{2}z^2+az+\frac{b}{2}\right)^2}.$$

The roots of the denominator are

$$z_1=\frac{-a+\sqrt{a^2-b^2}}{b}\quad\text{and}\quad z_2=\frac{-a-\sqrt{a^2-b^2}}{b}.$$

The assumption that $0<b<a$ implies that the only pole of f in the unit circle is at z_1. We must now compute the residue of f at z_1. We have

$$f(z)=\frac{z}{i\frac{b^2}{4}(z-z_1)^2(z-z_2)^2},$$

so the residue we are looking for is equal to the coefficient of the term $z-z_1$ in the power series expansion of

$$h(z)=\frac{4z}{ib^2(z-z_2)^2}.$$

Differentiating h once we find

$$h'(z)=\frac{4}{ib^2}\left[\frac{-z-z_2}{(z-z_2)^3}\right]$$

which evaluated at z_1 gives

$$\frac{4}{ib^2}\left[\frac{2a/b}{8(\sqrt{a^2-b^2})^3/b^3}\right]=\frac{a}{i(a^2-b^2)^{3/2}},$$

which is the residue of f at z_1. Thus

$$\int_C f(z)dz=2\pi i\frac{a}{i(a^2-b^2)^{3/2}}=\frac{2\pi a}{(a^2-b^2)^{3/2}}.$$

Exercise VI.2.24. *Let n be an even integer. Find*

$$\int_0^{2\pi}(\cos\theta)^n d\theta$$

by the method of residues.

Solution. We apply Theorem 2.3 with

$$f(z) = \frac{1}{2^n i z}\left(z + \frac{1}{z}\right)^n.$$

The only pole of f is at the origin. To find the residue of f at 0, we must find the constant term of $\left(z + \frac{1}{z}\right)^n$. Since n is even, the constant term is given by the binomial coefficient

$$\binom{n}{n/2} = \frac{n!}{(n/2)!(n - n/2)!} = \frac{n!}{(n/2)!^2},$$

and therefore, the residue of f at 0 is

$$\frac{n!}{2^n i (n/2)!^2}.$$

Hence

$$\int_0^{2\pi} (\cos\theta)^n d\theta = 2\pi i \frac{n!}{2^n i (n/2)!^2} = \frac{2\pi n!}{2^n (n/2)!^2}.$$